

A SIGNATURE FORMULA FOR HYPERELLIPTIC BROKEN LEFSCHETZ FIBRATIONS

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ABSTRACT. A hyperelliptic broken Lefschetz fibration is a generalization of a hyperelliptic Lefschetz fibration. We construct and compute a local signature of hyperelliptic directed broken Lefschetz fibrations by generalizing Endo's local signature of hyperelliptic Lefschetz fibrations. It is described by his local signature and a rational-valued homomorphism on the subgroup of the hyperelliptic mapping class group which preserves a simple closed curve setwise.

1. INTRODUCTION

A broken Lefschetz fibration is a smooth map introduced in [3] from a four-manifold to a surface which has at most two types of singularities, called Lefschetz singularity and indefinite fold singularity. It can be considered as a generalization of a Lefschetz fibration, and combining the results of Williams [18] and Lekili [12], it is proved that every closed oriented four-manifold admits a directed (more strictly, simplified) broken Lefschetz fibration.

In [10], we defined a hyperelliptic directed broken Lefschetz fibration as a generalization of a hyperelliptic Lefschetz fibration. We showed that, when the genus of any component of any fiber is greater than or equal to two, after blowing up several times, the total space is a double branched covering of a manifold obtained by blowing up a sphere bundle over the sphere. We also proved that the second rational homology class represented by a general fiber is nontrivial. As a corollary, $\sharp n\mathbb{CP}^2$ does not admit this fibration structure.

The purpose of this paper is to investigate the homeomorphism types of hyperelliptic directed broken Lefschetz fibrations generalizing Endo's local signature in [7] of hyperelliptic Lefschetz fibrations. See, for example, [1] and [2] for the history of local signatures.

We call a simple closed curve in Σ_g is type I or type II_h if it is non-separating or separating which bounds subsurfaces of genus h and $g - h$, respectively. We also call a Lefschetz singular fiber is type I

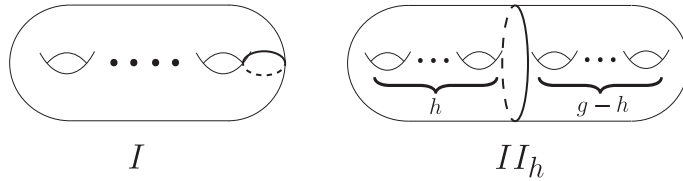


FIGURE 1. type I and type II_h

or type II_h if the vanishing cycle is type I or type II_h , respectively. Assign rational numbers to these types of singular fibers as

$$\sigma_{\text{loc}}(I) = -\frac{g+1}{2g+1}, \quad \sigma_{\text{loc}}(II_h) = \frac{4h(g-h)}{2g+1} - 1.$$

Endo showed that the signature of the total space of a hyperelliptic Lefschetz fibration is equal to the sum of these numbers of the Lefschetz singular fibers in the fibration (see Section 2.4, for details).

To explain our main theorem, we need some notation. Let Σ_g be a closed oriented surface of genus g , and \mathcal{M}_g denote its mapping class group. For a simple closed curve c in Σ_g , we denote by $\mathcal{M}_g(c)$ the subgroup of \mathcal{M}_g which consists of mapping classes represented by diffeomorphism preserving the curve c setwise. We also denote by $\mathcal{M}_g(c^{\text{ori}})$ the subgroup of $\mathcal{M}_g(c)$ which consists of mapping classes preserving the curve c setwise and its orientation. Let ι_g denote the involution of Σ_g as in Figure 1. We denote by \mathcal{H}_g the hyperelliptic mapping class group, that is, the subgroup of \mathcal{M}_g which consists

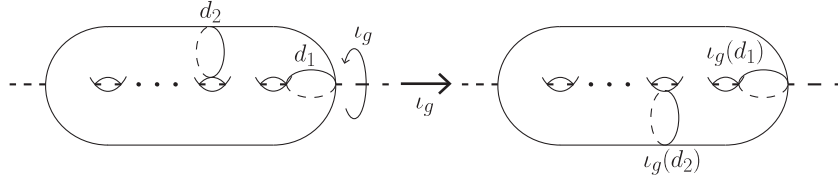


FIGURE 2. involution ι_g

of mapping classes represented by diffeomorphisms T satisfying $T\iota_g = \iota_g T$. For a simple closed curve c such that $\iota_g(c) = c$, we also denote by $\mathcal{H}_g(c)$ and $\mathcal{H}_g(c^{\text{ori}})$ the subgroups $\mathcal{H}_g(c) = \mathcal{H}_g \cap \mathcal{M}_g(c)$ and $\mathcal{H}_g(c^{\text{ori}}) = \mathcal{H}_g \cap \mathcal{M}_g(c^{\text{ori}})$, respectively. In Lemma 4.6, we will define rational-valued homomorphisms $h_{g,c}$ on $\mathcal{H}_g(c)$ when c is non-separating, and on $\mathcal{H}_g(c^{\text{ori}})$ when c is separating.

Let $f : M \rightarrow S^2$ be a directed broken Lefschetz fibration. We denote by Z_i the image of each component of the indefinite fold singularities under f . Decompose the 2-sphere into annuli A_i each of which is a neighborhood of $Z_i \subset S^2$ for $i = 1, 2, \dots, m$, and disks D_l and D_h as in Figure 3. We may choose D_h so that the image $\{y_1, \dots, y_n\} \subset S^2$ of all the Lefschetz singularities is in $\text{Int } D_h$. We denote by $\partial_1 A_i$ the boundary component of A_i such that $\partial_1 A_i = A_i \cap A_{i+1}$ for $i = 1, \dots, m-1$ and $\partial_1 A_m = A_m \cap D_l$. We also denote by $\partial_0 A_i$ the other boundary component of A_i .

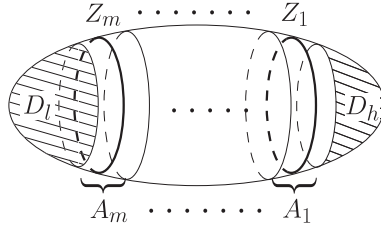


FIGURE 3. annuli A_i and disks D_l and D_h in S^2

For each $i = 1, \dots, m$, there is a unique component M_i of $f^{-1}(A_i)$ where an indefinite fold singularity exists. Let g_i denote the genus of a fiber in the mapping torus $\partial_0 A_i \cap M_i$. Identifying the fiber with Σ_{g_i} , we consider the vanishing cycle d_i of the indefinite fold singularity in Σ_{g_i} . We assume f to be hyperelliptic, whose definition we will give in Section 2.3. By the definition of the hyperelliptic directed broken Lefschetz fibration and Lemma 4.1 by Baykur, the monodromy φ_i of the mapping torus is in $\mathcal{H}_{g_i}(d_i)$. Then, our main theorem is as follows:

Theorem 1.1. *Let $f : M \rightarrow S^2$ be a hyperelliptic directed broken Lefschetz fibration as above. Then, we have*

$$\text{Sign } M = \sum_{i=1}^m h_{g_i, d_i}(\varphi_i) + \sum_{j=1}^n \sigma_{\text{loc}}(f^{-1}(y_j)).$$

We will prove Theorem 1.1 in Section 4.4. As we will see in Section 4.2, it is easy to calculate the explicit values of h_{g_i, d_i} since it is a homomorphism.

In Section 3, we will compute the abelianization and find a generating set of the groups $\mathcal{H}_g(c)$ and $\mathcal{H}_g(c^{\text{ori}})$. Let $c_1, c_2, \dots, c_{2g+1}$ be simple closed curves in Figure 4, and t_c denote the Dehn twist along a simple closed curve $c \subset \Sigma_g$.

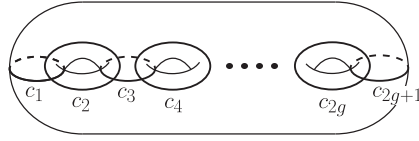


FIGURE 4. simple closed curves c_1, \dots, c_{2g+1}

Proposition 1.2. *Let $g \geq 1$.*

- (i) *Let c be a non-separating simple closed curve of type I in Figure 1. The group $\mathcal{H}_g(c)$ is generated by $\{t_{c_1}, \dots, t_{c_{2g-1}}, t_{c_{2g+1}}, t_g\}$.*
- (ii) *Let $1 \leq h \leq g-1$, and c a separating simple closed curve of type II_h in Figure 1. The group $\mathcal{H}_g(c^{\text{ori}})$ is generated by $\{t_{c_1}, t_{c_2}, \dots, t_{c_{2h}}, t_{c_{2h+2}}, t_{c_{2h+3}}, \dots, t_{c_{2g+1}}\}$.*

In Section 4, we will construct rational-valued homomorphisms $h_{g,c}$ on $\mathcal{H}_g(c)$ when c is non-separating, and on $\mathcal{H}_g(c^{\text{ori}})$ when c is separating. We will also compute their values on the generating sets in Proposition 1.2.

Proposition 1.3. (i) *Let $g \geq 1$, and c a non-separating simple closed curve of type I in Figure 1. The values of the homomorphism $h_{g,c} : \mathcal{H}_g(c) \rightarrow \mathbb{Q}$ are*

$$h_{g,c}(t_g) = 0, \quad h_{g,c}(t_{c_i}) = -\frac{1}{4g^2 - 1} \quad \text{for } i = 1, \dots, 2g-1, \quad \text{and } h_{g,c}(t_{c_{2g+1}}) = -\frac{g}{2g+1}.$$

- (ii) *Let $g \geq 1$, $0 \leq h \leq g$, and c a separating simple closed curve of type II_h in Figure 1. When $1 \leq h \leq g-1$, the values of the homomorphism $h_{g,c} : \mathcal{H}_g(c^{\text{ori}}) \rightarrow \mathbb{Q}$ are*

$$h_{g,c}(t_{c_i}) = \frac{g+1}{2g+1} - \frac{h+1}{2h+1} \quad \text{for } i = 1, \dots, 2h,$$

$$h_{g,c}(t_{c_i}) = \frac{g+1}{2g+1} - \frac{g-h+1}{2(g-h)+1} \quad \text{for } i = 2h+2, \dots, 2g.$$

When $h = 0, g$, the homomorphism $h_{g,c}$ is the zero map.

In Section 5, we will give examples of calculations of the signatures of simplified broken Lefschetz fibrations, and determine their homeomorphism types.

2. PRELIMINARY

2.1. Broken Lefschetz fibrations.

Definition 2.1. Let M and Σ be compact oriented smooth manifolds of dimension 4 and 2, respectively. A smooth map $f : M \rightarrow \Sigma$ is called a *broken Lefschetz fibration* (BLF, for short) if it satisfies the following conditions:

- (i) $f^{-1}(\partial\Sigma) = \partial M$
- (ii) f has at most the two types of singularities which is locally written as follows:
 - $(z_1, z_2) \mapsto \xi = z_1 z_2$, where (z_1, z_2) (resp. ξ) is a complex local coordinate of M (resp. Σ) compatible with its orientation;
 - $(t, x_1, x_2, x_3) \mapsto (y_1, y_2) = (t, x_1^2 + x_2^2 - x_3^2)$, where (t, x_1, x_2, x_3) (resp. (y_1, y_2)) is a real coordinate of M (resp. Σ).

The first singularity in the condition (ii) of Definition 2.1 is called a *Lefschetz singularity* and the second one is called an *indefinite fold singularity*. We denote by \mathcal{C}_f the set of Lefschetz singularities of f and by Z_f the set of indefinite fold singularities of f . We remark that a Lefschetz fibration is a BLF which has no indefinite fold singularities.

Let $f : M \rightarrow S^2$ be a BLF over the 2-sphere. Suppose that the restriction of f to the set of singularities is injective and that the image $f(Z_f)$ is the disjoint union of embedded circles parallel to the equator of S^2 . We put $f(Z_f) = Z_1 \amalg \cdots \amalg Z_m$, where Z_i is the embedded circle in S^2 . We choose a path $\alpha : [0, 1] \rightarrow S^2$ satisfying the following properties:

- (i) $\text{Im } \alpha$ is contained in the complement of $f(\mathcal{C}_f)$;
- (ii) α starts at the north pole $p_h \in S^2$, and ends at the south pole $p_l \in S^2$;
- (iii) α intersects each component of $f(Z_f)$ at one point transversely.

We put $\{q_i\} = Z_i \cap \text{Im } \alpha$ and $\alpha(t_i) = q_i$. We assume that q_1, \dots, q_m appear in this order when we go along α from p_h to p_l (see Figure 5). The preimage $f^{-1}(\text{Im } \alpha)$ is a 3-manifold which is a cobordism

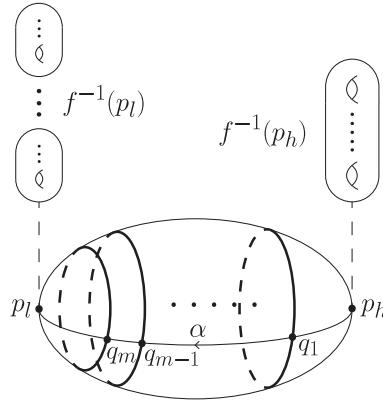


FIGURE 5. The example of the path α . The bold circles describe $f(Z_f)$.

between $f^{-1}(p_h)$ and $f^{-1}(p_l)$. By the local coordinate description of the indefinite fold singularity, it is easy to see that $f^{-1}(\alpha([0, t_i + \epsilon]))$ is obtained from $f^{-1}(\alpha([0, t_i - \epsilon]))$ by either 1 or 2-handle attachment for each $i = 1, \dots, m$. In particular, we obtain a handle decomposition of the cobordism $f^{-1}(\text{Im } \alpha)$.

Definition 2.2. A BLF $f : M \rightarrow S^2$ is said to be *directed* if it satisfies the following conditions:

- (i) the restriction of f to the set of singularities is injective and the image $f(Z_f)$ is the disjoint union of embedded circles parallel to the equator of S^2 ;
- (ii) all the handles in the above handle decomposition of $f^{-1}(\text{Im } \alpha)$ is index-2;
- (iii) all Lefschetz singularities of f are in the preimage of the component of $S^2 \setminus (Z_1 \amalg \cdots \amalg Z_m)$ which contains the point p_h .

We put $\{r_i\} = \partial_0 A_i \cap \text{Im } \alpha$ for $i = 1, \dots, m$ and $\{r_{m+1}\} = \partial_1 A_m \cap \text{Im } \alpha$. Using the path α , we can identify $f^{-1}(\alpha(t_i - \epsilon))$ with $f^{-1}(r_i)$. We call the attaching circle in $f^{-1}(r_i)$ of the 2-handle in Definition 2.2 the *vanishing cycle* of Z_i . A BLF is called *simplified* if it is directed, $m = 1$, and the vanishing cycle of Z_1 is non-separating.

2.2. Monodromy representations and vanishing cycles of Lefschetz singularities. Let $M \rightarrow \Sigma$ be an oriented surface bundle over a 2-manifold Σ . For a base point $y_0 \in \Sigma$, we denote by $\varrho : \pi_1(\Sigma, y_0) \rightarrow \mathcal{M}_g$ the monodromy representation. Let $f : M \rightarrow D^2$ be a Lefschetz fibration over a disk, and let $\mathcal{C}_f = \{z_1, \dots, z_n\}$ denote the set of Lefschetz singularities of f .

For each $i = 1, \dots, n$, put $y_i = f(z_i)$, and take an embedded path $\alpha_i : [0, 1] \rightarrow D^2$ satisfying

- each α_i connects y_0 to y_i ,
- $\alpha_i \cap f(\mathcal{C}_f) = \{y_i\}$,
- $\alpha_i \cap \alpha_j = \{y_0\}$ for all $i \neq j$,
- $\alpha_1, \dots, \alpha_n$ appear in this order when we travel counterclockwise around y_0 .

For each $i = 1, \dots, n$, we denote by $a_i \in \pi_1(D^2 \setminus f(\mathcal{C}_f), y_0)$ the element represented by the loop obtained by connecting a counterclockwise circle around y_i to y_0 by using α_i . The sequence $W_f = (\varrho_f(a_1), \dots, \varrho_f(a_n)) \in (\mathcal{M}_g)^n$ is called a *Hurwitz system* of f . By the conditions on paths a_1, \dots, a_n , the product $\varrho_f(a_1) \cdots \varrho_f(a_n)$ is equal to the monodromy along the boundary ∂D^2 . It is known that each $\varrho_f(a_i)$ is the right-handed Dehn twist along a certain simple closed curve c_i , called the *vanishing cycle* of the Lefschetz singularity z_i (see [11] or [14]).

2.3. The hyperelliptic mapping class group and hyperelliptic directed BLFs. Endow the relative topology with the centralizer $C(\iota_g)$ of ι_g in the diffeomorphism group $\text{Diff}_+ \Sigma_g$. The inclusion homomorphism $C(\iota_g) \rightarrow \text{Diff}_+ \Sigma_g$ induces a natural homomorphism $\pi_0 C(\iota_g) \rightarrow \mathcal{M}_g$ between their path-connected components. We denote this group $\pi_0 C(\iota_g)$ by \mathcal{H}_g^s . Birman and Hilden showed:

Theorem 2.3 (Birman-Hilden [5, Corollary 7.1]). *When $g \geq 2$, the homomorphism $\mathcal{H}_g^s \rightarrow \mathcal{M}_g$ is injective.*

The image of the above homomorphism is called the *hyperelliptic mapping class group*, and denoted by \mathcal{H}_g . Actually, they showed the above result for more general settings, but we only use the case for the involution $\iota_g : \Sigma_g \rightarrow \Sigma_g$. See [6], for more details.

Let $f : M \rightarrow S^2$ be a directed BLF. For $i = 1, \dots, m$, let $d_i \subset f^{-1}(r_i)$ denote the vanishing cycle of Z_i . Fix an identification $f^{-1}(p_h)$ with $\Sigma_{n_1} \amalg \cdots \amalg \Sigma_{n_k}$ for some integers n_1, \dots, n_k . Then, we can define an involution of $f^{-1}(p_h)$ by $\iota_{n_1} \amalg \cdots \amalg \iota_{n_k}$. By using the path α , we can identify $f^{-1}(p_h)$ with $f^{-1}(r_1)$ and $f^{-1}(r_i) \setminus \{\text{two points}\}$ with $f^{-1}(r_{i+1}) \setminus d_i$. Hence, we also obtain an involution of $f^{-1}(r_i)$ by the hyperelliptic involution of $f^{-1}(p_h)$ for $i = 1, \dots, m$.

Definition 2.4. A directed BLF $f : M \rightarrow S^2$ is said to be *hyperelliptic* if it satisfies the following conditions for a suitable identification of $f^{-1}(p_h)$ with $\Sigma_{n_1} \amalg \cdots \amalg \Sigma_{n_k}$:

- the image of the monodromy representation of the Lefschetz fibration $\text{res } f : f^{-1}(D_h) \rightarrow D_h$ is contained in the group \mathcal{H}_g ,
- d_i is preserved by the involution up to isotopy.

In the following, we review some properties of the hyperelliptic mapping class group. Let X be a 2-disk or a 2-sphere. For a positive integer n and distinct points $\{p_i\}_{i=1}^n$ in $\text{Int } X$, Denote by $\text{Diff}_+(X, \partial X, \{p_1, p_2, \dots, p_n\})$ the group defined by

$$\begin{aligned} & \text{Diff}_+(X, \partial X, \{p_1, p_2, \dots, p_n\}) \\ &= \{T \in \text{Diff}_+ X \mid T|_{\partial X} \text{ is the identity map, and } T(\{p_1, p_2, \dots, p_n\}) = \{p_1, p_2, \dots, p_n\}\}. \end{aligned}$$

Denote by \mathcal{M}_0^n or $\mathcal{M}_{0,1}^n$ its mapping class group when $X = S^2$ or $X = D^2$, respectively. Let D_i be a disk in $\text{Int } X$ which includes p_i and p_{i+1} but is disjoint from all p_j for $j \neq i, i+1$, and denote by $\nu(\partial D_i)$ a neighborhood of the boundary ∂D_i in D_i . Choose a diffeomorphism $T_i \in \text{Diff}_+(X, \partial X, \{p_1, p_2, \dots, p_n\})$ such that $T_i|_{D_i}$ interchanges the points p_i and p_{i+1} , $T_i|_{X - \text{Int } D_i}$ is the identity map, and T_i^2 is isotopic to the Dehn twist along ∂D_i (see Birman-Hilden p.87-88 for details). The mapping class group \mathcal{M}_0^n and $\mathcal{M}_{0,1}^n$ is generated by $\{\sigma_i\}_{i=1}^{n-1}$, where σ_i is the mapping class represented by the diffeomorphism T_i .

Identifying the quotient space $\Sigma_g / \langle \iota_g \rangle$ with S^2 , let $\{p_1, p_2, \dots, p_{2g+1}, p_{2g+2}\} \subset S^2$ be the branched set of the quotient map $\Sigma_g \rightarrow \Sigma_g / \langle \iota_g \rangle$. By the definition, any diffeomorphism T in $C(\iota_g)$ satisfies $T\iota_g(x) = \iota_g T(x)$ for $x \in \Sigma_g$. Hence, there exists a unique diffeomorphism $\bar{T} \in \text{Diff}_+ S^2$ such that the diagram

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{T} & \Sigma_g \\ p \downarrow & & \downarrow p \\ S^2 & \xrightarrow{\bar{T}} & S^2 \end{array}$$

commutes. Moreover, it satisfies $\bar{T}(\{p_1, p_2, \dots, p_{2g+2}\}) = \{p_1, p_2, \dots, p_{2g+2}\} \subset S^2$.

By the above diagram, we can define

$$\mathcal{P}_g : \mathcal{H}_g^s \rightarrow \mathcal{M}_0^{2g+2}$$

by $\mathcal{P}_g([T]) = [\bar{T}]$.

Theorem 2.5 (Birman-Hilden [5, Theorem 1]). *Let $g \geq 1$. the sequence*

$$1 \longrightarrow \langle \iota_g \rangle \longrightarrow \mathcal{H}_g^s \xrightarrow{\mathcal{P}_g} \mathcal{M}_0^{2g+2} \longrightarrow 1$$

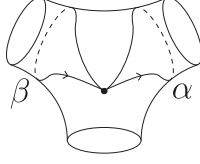
is exact.

They showed the homomorphism $\mathcal{P}_g : \mathcal{H}_g^s \rightarrow \mathcal{M}_0^{2g+2}$ maps the Dehn twist t_{c_i} to σ_i in [5, Theorem 2]. Furthermore, they proved:

Proposition 2.6. *Let $g \geq 1$. The group \mathcal{H}_g^s is generated by $\{t_{c_1}, \dots, t_{c_{2g+1}}\}$.*

2.4. Meyer's signature cocycle and the local signature for hyperelliptic Lefschetz fibrations. It is known that, for a hyperelliptic Lefschetz fibration $f : M \rightarrow \Sigma$ over a closed oriented surface Σ , the signature $\text{Sign } M$ is described as the sum of invariants of the singular fiber germs in M . We review this invariant.

Let φ, ψ be elements in the mapping class group \mathcal{M}_g . We denote by $E_{\varphi, \psi}$ a Σ_g -bundle over a pair of pants $S^2 - \Pi_{i=1}^3 \text{Int } D^2$ whose monodromies along α and β in Figure 6 are φ and ψ , respectively.

FIGURE 6. paths α and β

Theorem 2.7 (Meyer [15]). *Define a 2-cochain $\tau_g : \mathcal{M}_g \times \mathcal{M}_g \rightarrow \mathbb{Z}$ of the mapping class group by $\tau_g(\varphi, \psi) = -\text{Sign } E_{\varphi, \psi}$. Then, τ_g is a 2-cocycle, and the order of its homology class is as follows.*

- (i) *The order of $[\tau_1] \in H^2(\mathcal{M}_1; \mathbb{Z})$ is 3,*
- (ii) *The order of $[\tau_2] \in H^2(\mathcal{M}_2; \mathbb{Z})$ is 5,*
- (iii) *When $g \geq 3$, $[\tau_g] \neq 0 \in H^2(\mathcal{M}_g; \mathbb{Q})$.*

Proposition 2.8 (Endo [7]). *If we restrict τ_g to \mathcal{H}_g , the order of $[\tau_g] \in H^2(\mathcal{H}_g; \mathbb{Z})$ is $2g + 1$.*

Since τ_g represents a trivial homology class in $H^2(\mathcal{H}_g; \mathbb{Q})$, there exists a cobounding function $\phi_g : \mathcal{H}_g \rightarrow \mathbb{Q}$ of it. Furthermore, since $H_1(\mathcal{H}_g; \mathbb{Q})$ is trivial, this cobounding function ϕ_g is unique.

Lemma 2.9 (Endo [7, Proof of Theorem 4.4]). *Let $f : M \rightarrow \Sigma$ be a Σ_g -bundle over a compact oriented surface Σ . Assume that the image of the monodromy representation $\pi_1(\Sigma, y_0) \rightarrow \mathcal{M}_g$ is in \mathcal{H}_g if we choose a suitable identification $f^{-1}(y_0) \cong \Sigma_g$. Let $\{\partial_j\}_{j=1}^n$ denote the boundary components of Σ , and give orientations coming from Σ . Then, we have*

$$\text{Sign } M = - \sum_{j=1}^n \phi(\psi_j),$$

where $\psi_j \in \mathcal{H}_g$ is the monodromy along ∂_j .

Using this function, he generalized the local signature of Lefschetz fibrations of genus 1 [13] and of genus 2 [14] constructed by Matsumoto. Let $f : M \rightarrow \Sigma$ be a hyperelliptic Lefschetz fibration of genus g over a closed oriented surface Σ , and y_1, \dots, y_n the image of the set of Lefschetz singularities under f . For the Lefschetz singular fiber $f^{-1}(y_j)$, define a rational number $\sigma_{\text{loc}}(f^{-1}(y_j))$ by

$$\sigma_{\text{loc}}(f^{-1}(y_j)) = -\phi_g(\varphi_j) + \text{Sign}(f^{-1}\nu(y_j)),$$

where $\varphi_j \in \mathcal{H}_g$ is the monodromy along $\partial\nu(y_j)$. He computed the values for Lefschetz singular fibers as in Introduction, and showed:

Theorem 2.10 (Endo [7, Theorem 4.4]). *Let $f : M \rightarrow \Sigma$ be a hyperelliptic Lefschetz fibration as above. Then, we have*

$$\text{Sign } M = \sum_{i=1}^n \sigma_{\text{loc}}(f^{-1}(y_i)).$$

3. A SUBGROUP $\mathcal{H}_g(c)$ OF THE HYPERELLIPTIC MAPPING CLASS GROUP WHICH PRESERVES A CURVE c

In this section, we investigate the abelianization and a generating set of the hyperelliptic mapping class group $\mathcal{H}_g(c)$ which fix the curve c . In the last paragraphs of Section 3.1 and Section 3.2, we will prove Proposition 1.2.

Consider the case when c is nonseparating. If we take a diffeomorphism $T \in \text{Diff}_+ \Sigma_g$ which fixes the curve c setwise, it induces the diffeomorphism $\Sigma_g \setminus c \rightarrow \Sigma_g \setminus c$. This diffeomorphism can be extended to the diffeomorphism of \hat{T} of Σ_{g-1} by regarding $\Sigma_g \setminus c$ as the surface of genus $g-1$ with two punctures. Hence, we can define a homomorphism $\Phi_n : \mathcal{M}_g(c) \rightarrow \mathcal{M}_{g-1}$ by $\Phi_n([T]) = [\hat{T}]$. Next, consider the case when c is a separating curve in Σ_g bounding subsurfaces of genus h and $g-h$. Identifying $\Sigma_g \setminus c$ with disjoint sum of two punctured surfaces of genus h and $g-h$, we can also define a homomorphism $\Phi_s : \mathcal{M}_g(c^{\text{ori}}) \rightarrow \mathcal{M}_h \times \mathcal{M}_{g-h}$.

3.1. When c is non-separating. First, consider the case when c is type I. For simplicity, we choose c as in Figure 1. Let $\gamma \in \Sigma_g / \langle \iota_g \rangle$ be the projection of the curve c by $p : \Sigma_g \rightarrow \Sigma_g / \langle \iota_g \rangle$. Identifying $\Sigma_g / \langle \iota_g \rangle$ with S^2 , define a group $\mathcal{M}_0^{2g}(\gamma)$ by

$$\mathcal{M}_0^{2g}(\gamma) = \{[T] \in \mathcal{M}_0^{2g+2} \mid T(\gamma) = \gamma\}.$$

For a diffeomorphism $T \in C(\iota_g)$, we have a diffeomorphism $\bar{T} \in \text{Diff}_+(S^2, p_1, p_2, \dots, p_{2g+1}, p_{2g+2})$ defined by $pT = \bar{T}p$ as in Section 2.3. Moreover, if $T \in C(\iota_g)$ preserves c setwise, \bar{T} also preserves the path γ setwise. Hence, the image $\mathcal{P}_g(\mathcal{H}_g^s(c))$ is contained in $\mathcal{M}_0^{2g}(\gamma)$. Conversely, if $\bar{T} \in \text{Diff}_+(S^2, p_1, p_2, \dots, p_{2g+1}, p_{2g+2})$ preserves the path γ setwise, there is a diffeomorphism $T \in C(\iota_g)$ such that $T(c) = c$ and $pT = \bar{T}p$. Thus, we have $\mathcal{P}_g(\mathcal{H}_g^s(c)) = \mathcal{M}_0^{2g}(\gamma)$. Consider the exact sequence obtained by restricting the homomorphism $\mathcal{P}_g : \mathcal{H}_g^s \rightarrow \mathcal{M}_0^{2g+2}$ in Theorem 2.5 to $\mathcal{H}_g^s(c)$.

Lemma 3.1. *For $g \geq 1$, the exact sequence*

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathcal{H}_g^s(c) \xrightarrow{\mathcal{P}_g} \mathcal{M}_0^{2g}(\gamma) \longrightarrow 1$$

splits. In particular, we have $\mathcal{H}_g^s(c) \cong \mathbb{Z}/2\mathbb{Z} \times \mathcal{M}_0^{2g}(\gamma)$.

Proof. Define a map $\lambda : \mathcal{H}_g^s(c) \rightarrow \mathbb{Z}/2\mathbb{Z}$ by $\lambda(\varphi) = 0$ if $\varphi_*[c] = [c] \in H_1(\Sigma_g; \mathbb{Z})$, and $\lambda(\varphi) = 1$ if $\varphi_*[c] = -[c] \in H_1(\Sigma_g; \mathbb{Z})$. Then, λ is a homomorphism, and satisfies $\lambda([\iota_g]) = 1 \in \mathbb{Z}/2\mathbb{Z}$. Thus, it induces a splitting of the exact sequence. \square

Let $s : \partial D^2 \rightarrow \partial D^2$ denote the half-rotation of the circle. Let $\mathcal{M}_{0, \text{half}}^{2g}$ denote the group which consists of the path-connected components of $\{T \in \text{Diff}_+(D^2, p_1, p_2, \dots, p_{2g}) \mid T|_{\partial D^2} = s \text{ or } \text{id}_{\partial D^2}\}$.

Lemma 3.2. *Let $g \geq 1$.*

$$\mathcal{M}_0^{2g}(\gamma) \cong \mathcal{M}_{0, \text{half}}^{2g}.$$

Proof. Let $\mathcal{M}_0^{2g}(\gamma^{\text{ori}})$ be a subgroup of $\mathcal{M}_0^{2g}(\gamma)$ consists of mapping classes which preserve the orientation of the path γ . First, we prove the isomorphism

$$\mathcal{M}_0^{2g}(\gamma^{\text{ori}}) \cong \mathcal{M}_{0,1}^{2g}.$$

Let $\text{Diff}_+(S^2, \{p_1, \dots, p_{2g+2}\}, [\gamma])$ be the group consists of orientation-preserving diffeomorphisms $T : S^2 \rightarrow S^2$ such that $T(\{p_1, \dots, p_{2g+2}\}) = \{p_1, \dots, p_{2g+2}\}$ and there exists a closed neighborhood $\nu(\gamma)$ of γ where $T|_{\nu(\gamma)}$ is the identity map. Let T be a representative of a mapping class in $\mathcal{M}_0^{2g}(\gamma^{\text{ori}})$. Using the isotopy extension theorem, we can change T into a diffeomorphism in $\text{Diff}_+(S^2, \{p_1, \dots, p_{2g+2}\}, [\gamma])$ by some isotopy. Moreover, we can also prove that

$$\mathcal{M}_0^{2g}(\gamma^{\text{ori}}) \cong \pi_0 \text{Diff}_+(S^2, \{p_1, \dots, p_{2g+2}\}, [\gamma]),$$

using the isotopy extension theorem. Similarly, let $\text{Diff}_+(S^2 - \text{Int } D^2, p_1, \dots, p_{2g}, [\partial D^2])$ be a group consists of orientation-preserving diffeomorphisms $T : S^2 - \text{Int } D^2 \rightarrow S^2 - \text{Int } D^2$ such that there exists a closed neighborhood $\nu(\partial D^2)$ where $T|_{\nu(\partial D^2)}$ is the identity map. We can also show that

$$\mathcal{M}_{0,1}^{2g} \cong \pi_0 \text{Diff}_+(S^2 - \text{Int } D^2, p_1, \dots, p_{2g}, [\partial D^2]).$$

Separate the circle ∂D^2 into two arcs $\alpha : [0, 1] \rightarrow \partial D^2$ and $\beta : [0, 1] \rightarrow \partial D^2$ such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. If we identify $\alpha(t)$ and $\beta(t)$ in $S^2 - \text{Int } D^2$, the quotient space is diffeomorphic to S^2 . Choose an identification L of the $(2g + 3)$ -tuples

$$(S^2 - \text{Int } D^2 / (\alpha(t) \sim \beta(t)), p_1, \dots, p_{2g}, \alpha(0), \alpha(1)) \cong (S^2, p_1, \dots, p_{2g}, p_{2g+1}, p_{2g+2}).$$

Since a diffeomorphism $T \in \text{Diff}_+(S^2 - \text{Int } D^2)$ satisfying $T|_{\nu(\partial D^2)} = \text{id}_{\nu(\partial D^2)}$ induces a diffeomorphism \bar{T} of $S^2 - \text{Int } D^2 / (\alpha(t) \sim \beta(t))$, we have the isomorphism $\mathcal{M}_{0,1}^{2g} \cong \mathcal{M}^{2g}(\gamma^{\text{ori}})$ defined by $[T] \mapsto [L\bar{T}L^{-1}]$.

Next, we prove $\mathcal{M}_0^{2g}(\gamma) \cong \mathcal{M}_{0,\text{half}}^{2g}$. Choose a diffeomorphism $r \in \text{Diff}_+(S^2 - \text{Int } D^2)$ such that $r\alpha(t) = \beta(1-t)$ and $r(\{p_1, \dots, p_{2g}\}) = \{p_1, \dots, p_{2g}\}$. It induces a diffeomorphism $\bar{r} \in \text{Diff}_+ S^2$ such that $\bar{r}(\{p_1, \dots, p_{2g}\}) = \{p_1, \dots, p_{2g}\}$, $\bar{r}(p_{2g+1}) = p_{2g+2}$, and $\bar{r}(p_{2g+2}) = p_{2g+1}$. Consider the group consisting of diffeomorphisms T of S^2 such that $T(\{p_1, \dots, p_{2g+2}\}) = \{p_1, \dots, p_{2g+2}\}$, and $T|_{\nu(\gamma)}$ is equal to $\bar{r}|_{\nu(\gamma)}$ or $\text{id}_{\nu(\gamma)}$ for some closed neighborhood $\nu(\gamma)$ instead of $\text{Diff}_+(S^2, \{p_1, \dots, p_{2g+2}\}, [\gamma])$. In the same way, consider the group consisting of diffeomorphisms T of $S^2 - \text{Int } D^2$ such that $T(\{p_1, \dots, p_{2g}\}) = \{p_1, \dots, p_{2g}\}$, and $T|_{\nu(\partial D^2)}$ is equal to $r|_{\nu(\partial D^2)}$ or $\text{id}_{\nu(\partial D^2)}$ instead of $\text{Diff}_+(S^2 - \text{Int } D^2, p_1, \dots, p_{2g}, [\partial D^2])$. Then, we have the isomorphism between their path-connected components, similarly. Thus, we have $\mathcal{M}^{2g}(\gamma) \cong \mathcal{M}_{0,\text{half}}^{2g}$. \square

We can define a homomorphism $\mathcal{M}_{0,\text{half}}^{2g} \rightarrow \langle s \rangle$ by mapping $[T]$ to $T|_{\partial D^2}$, where $\langle s \rangle$ is the cyclic group of order 2 generated by s . Then, the kernel is the subgroup $\mathcal{M}_{0,1}^{2g}$.

Lemma 3.3. *For $g \geq 1$, the exact sequence*

$$1 \longrightarrow \mathcal{M}_{0,1}^{2g} \longrightarrow \mathcal{M}_{0,\text{half}}^{2g} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

splits.

Proof. We may assume p_1, \dots, p_{2g} are arranged in the disk as in Figure 7. Consider an involution $\mu \in \text{Diff}_+(D^2, p_1, \dots, p_{2g})$ which rotates the disk 180 degrees and interchanges the points p_i and p_{g+i} for $i = 1, \dots, g$. Define a homomorphism $j : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathcal{M}_{0,\text{half}}^{2g}$ by $j(1) = \mu$. This induces the splitting

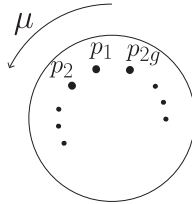


FIGURE 7. p_1, \dots, p_{2g} in D^2

of the above exact sequence. \square

Lemma 3.4. *Let $g \geq 1$, and c a non-separating simple closed curve such that $\iota_g(c) = c$. Then, we have*

$$H_1(\mathcal{H}_g^s(c); \mathbb{Z}) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2.$$

Proof. By Lemma 3.1 and Lemma 3.3, we have

$$H_1(\mathcal{H}_g^s(c); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus H_1(\mathcal{M}_0^{2g}(\gamma); \mathbb{Z}), \text{ and } H_1(\mathcal{M}_{0,\text{half}}^{2g}; \mathbb{Z}) \cong H_1(\mathcal{M}_{0,1}^{2g}; \mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

We showed $\mathcal{M}_0^{2g}(\gamma) \cong \mathcal{M}_{0,\text{half}}^{2g}$ in Lemma 3.2, and it is known that $H_1(\mathcal{M}_{0,1}^{2g}; \mathbb{Z}) \cong \mathbb{Z}$ (see, for example, [8, Section 9.1.3 and 9.2]). Hence, we have $H_1(\mathcal{H}_g^s(c); \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$. \square

Define a group $\mathcal{H}_g^s(c)$ by $\mathcal{H}_g^s(c) = \{[T] \in \mathcal{H}_g^s \mid T(c) = c\}$. If we restrict the homomorphism $\mathcal{H}_g^s \rightarrow \mathcal{H}_g$ in Theorem 2.3 to $\mathcal{H}_g^s(c)$, we have a homomorphism $\mathcal{H}_g^s(c) \rightarrow \mathcal{H}_g(c)$. Note that it is not obvious that this homomorphism is surjective, in other words, mapping classes in $\mathcal{H}_g(c)$ can be represented by elements in $C(\iota_g)$ which fix the curve c setwise. In [10, Lemma 3.1], we showed:

Lemma 3.5. *Let $g \geq 1$, and c an essential simple closed curve in Σ_g . The homomorphism*

$$\mathcal{H}_g^s(c) \rightarrow \mathcal{H}_g(c)$$

is surjective.

By Theorem 2.3, this is also injective when $g \geq 2$.

Consider the case when $g = 1$. As is well-known, the group \mathcal{H}_1 coincides with \mathcal{M}_1 . Hence, $\mathcal{H}_1(c)$ also coincides with $\mathcal{M}_1(c)$. If $c = c_3$ in Figure 4, the group $\mathcal{M}_1(c)$ is described as

$$\mathcal{M}_1(c) = \left\{ \begin{pmatrix} \epsilon & n \\ 0 & \epsilon \end{pmatrix} \in \text{SL}(2; \mathbb{Z}) \mid \epsilon \in \{\pm 1\}, n \in \mathbb{Z} \right\}.$$

By mapping $[T] \in \mathcal{M}_1(c)$ to $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, we have a split exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_1(c) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

Thus, we have $H_1(\mathcal{H}_1(c); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Combining Lemma 3.4, Lemma 3.5, and the case when $g = 1$ as above, we have:

Lemma 3.6. *Let c be a non-separating simple closed curve such that $\iota_g(c) = c$. Then, we have*

$$H_1(\mathcal{H}_g(c); \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{when } g \geq 2, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{when } g = 1. \end{cases}$$

Proof of Proposition 1.2 (i). Let $\sigma \in \mathcal{M}_{0,\text{half}}^{2g}$ denote the half twist along ∂D^2 . By the exact sequence in Lemma 3.3, the group $\mathcal{M}_{0,\text{half}}^{2g}$ is generated by $\{\sigma_1, \dots, \sigma_{2g-1}, \sigma\}$. By [5, Theorem 2], we have $\mathcal{P}_g(t_{c_i}) = \sigma_i$ for $i = 1, \dots, 2g$ and $\mathcal{P}_g(t_{c_{2g+1}}) = \sigma$. By the exact sequence in Lemma 3.1, the group $\mathcal{H}_g(c)$ is generated by t_{c_i} for $i = 1, 2, \dots, 2g-1, 2g+1$ and ι_g . \square

3.2. When c is separating. Next, consider the case when c is type II_h . For simplicity, we choose c as in Figure 1.

As we will see in Section 4.1, when the vanishing cycle of Z_i in the hyperelliptic directed BLF is separating, the image of the monodromy representation along $\partial_0 A_i$ is contained in $\mathcal{H}_g(c^{\text{ori}})$. Hence, we only consider the group $\mathcal{H}_g(c^{\text{ori}})$ in this section instead of $\mathcal{H}_g(c)$. Of course, if $g \neq 2h$, we have $\mathcal{H}_g(c) = \mathcal{H}_g(c^{\text{ori}})$ since any diffeomorphism of Σ_g which preserves c setwise acts trivially on $\pi_0(\Sigma_g - c)$.

First, consider the case when $h = 0, g$. For any diffeomorphism T of Σ_g , we can change T so that it preserves c setwise by some isotopy. Thus, we have $\mathcal{H}_g(c^{\text{ori}}) = \mathcal{H}_g$.

In the following, we only consider the case $1 \leq h \leq g - 1$. Choose a disk D in $\Sigma_g - \bigcup_{i=1}^{2g} c_i$ so that $\iota_g(D) = D$, where c_i is the simple closed curve in Figure 4. Denote by $\Sigma_{g,1}$ the subsurface $\Sigma_g - \text{Int } D$, and by $\iota_{g,1}$ the restriction of ι_g to $\Sigma_{g,1}$. The mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$ is defined by $\mathcal{M}_{g,1} = \pi_0 \text{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1})$, where $\text{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1})$ is the diffeomorphism group of $\Sigma_{g,1}$ with C^∞ topology which fixes the boundary pointwise.

We identify the subsurfaces of Σ_g bounded by c with $\Sigma_{h,1}$ and $\Sigma_{g-h,1}$ so that $\iota_g|_{\Sigma_{h,1}} = \iota_{h,1}$ and $\iota_g|_{\Sigma_{g-h,1}} = \iota_{g-h,1}$. For $T_1 \in \text{Diff}_+(\Sigma_{h,1}, \partial\Sigma_{h,1})$ and $T_2 \in \text{Diff}_+(\Sigma_{g-h,1}, \partial\Sigma_{g-h,1})$, the diffeomorphism $T_1 \cup T_2 \in \text{Diff}_+\Sigma_g$ preserves the curve c . Hence, we can define a map

$$\Psi : \mathcal{M}_{h,1} \times \mathcal{M}_{g-h,1} \rightarrow \mathcal{M}_g(c^{\text{ori}})$$

by $\Psi([T_1], [T_2]) = [T_1 \cup T_2]$. This is a well-defined homomorphism.

Define a subgroup $\mathcal{H}_{g,1}$ of $\mathcal{M}_{g,1}$ by $\mathcal{H}_{g,1} = \{[T] \in \mathcal{M}_{g,1} \mid \iota_{g,1} T \iota_{g,1}^{-1} = T\}$. Apparently, the image $\Psi(\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1})$ is contained in the subgroup $\mathcal{H}_g(c^{\text{ori}}) \subset \mathcal{M}_g(c^{\text{ori}})$.

Lemma 3.7. *Let $g \geq 2$. When $1 \leq h \leq g - 1$, the sequence*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1} \xrightarrow{\Psi} \mathcal{H}_g(c^{\text{ori}}) \longrightarrow 1$$

is exact.

Proof. By [8, Theorem 3.18], we have

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{h,1} \times \mathcal{M}_{g-h,1} \xrightarrow{\Psi} \mathcal{M}_g(c^{\text{ori}}) \longrightarrow 1.$$

The kernel of Ψ is generated by $(t_{\partial\Sigma_1}, t_{\partial\Sigma_2}^{-1})$, and it is contained in $\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1}$. Thus, we only need to prove $\Psi(\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1}) = \mathcal{H}_g(c^{\text{ori}})$.

Let φ be a mapping class in $\mathcal{H}_g(c)$. By Lemma 3.5, we can choose a representative $T \in \text{Diff}_+\Sigma_g$ of φ satisfying $T\iota_g = \iota_g T$ and $T(c) = c$. Using some isotopy, we may assume $T|_c$ is the identity map. Then, $T|_{\Sigma_{h,1}}$ and $T|_{\Sigma_{g-h,1}}$ represent mapping classes in $\mathcal{H}_{h,1}$ and $\mathcal{H}_{g-h,1}$, respectively. Since $\Psi([T|_{\Sigma_{h,1}}], [T|_{\Sigma_{g-h,1}}]) = [T]$, we obtain $\Psi(\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1}) = \mathcal{H}_g(c^{\text{ori}})$. \square

Let $C(\iota_{g,1})$ be the group defined by $C(\iota_{g,1}) = \{T \in \text{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1}) \mid \iota_{g,1} T \iota_{g,1}^{-1} = T\}$. We have the homomorphism $\mathcal{P}_{g,1} : \pi_0(C(\iota_{g,1})) \rightarrow \mathcal{M}_{0,1}^{2g+1}$ defined by $[T] \mapsto [\tilde{T}]$ in the same way as $\mathcal{P}_g : \mathcal{H}_g^s \rightarrow \mathcal{M}_0^{2g+2}$ in Section 2.3. Since any isotopy of $\text{Diff}_+(D^2, \partial D^2, \{p_1, \dots, p_{2g+1}\})$ can be lifted to an isotopy of $C(\iota_{g,1})$, $\text{Ker}(\mathcal{P}_{g,1})$ is represented by the deck transformation $\iota_{g,1}$ or $\text{id}_{\Sigma_{g,1}}$. Since $C(\iota_{g,1})$ does not contain $\iota_{g,1}$, the kernel of the homomorphism $\mathcal{P}_{g,1}$ is trivial. Furthermore, $\mathcal{P}_{g,1} : \pi_0 C(\iota_{g,1}) \rightarrow \mathcal{M}_{0,1}^{2g+1}$ is an isomorphism since $\mathcal{M}_{0,1}^{2g+1}$ is generated by $\{\sigma_i\}_{i=1}^{2g}$ and $\mathcal{P}_{g,1}(t_{c_i}) = \sigma_i$ for $i = 1, \dots, 2g$.

Lemma 3.8. *For $g \geq 1$, the natural homomorphism $\pi_0 C(\iota_{g,1}) \rightarrow \mathcal{H}_{g,1}$ is an isomorphism.*

Proof. By the definition of $\mathcal{H}_{g,1}$, the natural homomorphism $\pi_0(C(\iota_{g,1})) \rightarrow \mathcal{H}_{g,1}$ is surjective. Hence, it suffices to show the injectivity.

Embed $\Sigma_{g,1}$ in Σ_{g+1} so that $\iota_{g+1}|_{\Sigma_{g,1}} = \iota_{g,1}$. For a diffeomorphism T of $\Sigma_{g,1}$, we can extend T to a diffeomorphism \tilde{T} of Σ_{g+1} by the identity map on $\Sigma_{g+1} \setminus \Sigma_{g,1}$. Thus, we have homomorphisms $\pi_0(C(\iota_{g,1})) \rightarrow \pi_0(C(\iota_{g+1}))$ and $\mathcal{H}_{g,1} \rightarrow \mathcal{H}_{g+1}$ defined by $[T] \mapsto [\tilde{T}]$. By gluing a disk with three marked points to D^2 , we can also define a homomorphism $\mathcal{M}_{0,1}^{2g+1} \rightarrow \mathcal{M}_0^{2g+4}$ in the same way. By in [8, Theorem 3.18], the latter homomorphism is injective.

If we consider $(\Sigma_{g+1} \setminus \text{Int } \Sigma_{g,1}) / \langle \iota_{g+1} \rangle$ as a disk with three marked points, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_{0,1}^{2g+1} & \xleftarrow{\mathcal{P}_{g,1}} & \pi_0 C(\iota_{g,1}) & \longrightarrow & \mathcal{H}_{g,1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_0^{2g+4} & \xleftarrow{\mathcal{P}_{g+1}} & \pi_0 C(\iota_{g+1}) & \longrightarrow & \mathcal{H}_{g+1}. \end{array}$$

The left side shows that $\pi_0 C(\iota_{g,1}) \rightarrow \pi_0 C(\iota_{g+1})$ is injective. By Theorem 2.3, the right side shows that $\pi_0 C(\iota_{g,1}) \rightarrow \mathcal{H}_{g,1}$ is also injective. \square

Lemma 3.9. *Let $g \geq 2$ and $1 \leq h \leq g-1$. Let c be a separating simple closed curve which bounds subsurfaces of genus h and $g-h$ and satisfies $\iota_g(c) = c$. Then, we have*

$$H_1(\mathcal{H}_g(c^{\text{ori}}); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z},$$

where $d = \gcd(4h(2h+1), 4(g-h)(2g-2h+1))$.

Proof. Since $\mathcal{H}_{h,1} \cong \mathcal{M}_{0,1}^{2h+1}$, we have $H_1(\mathcal{H}_{h,1}; \mathbb{Z}) \cong \mathbb{Z}$. By the chain relation (see, for example, in [8, Proposition 4.12]), the mapping class $(t_{c_1} \cdots t_{c_{2h}})^{4h+2} \in \mathcal{H}_{h,1}$ coincides with the Dehn twist $t_{\partial \Sigma_{h,1}}$ along the boundary. In the same way, we have $(t_{c_1} \cdots t_{c_{2(g-h)}})^{4(g-h)+2} = t_{\partial \Sigma_{g-h,1}} \in \mathcal{H}_{g-h,1}$.

The kernel of the homomorphism $\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1} \rightarrow \mathcal{H}_g(c^{\text{ori}})$ is the cyclic group generated by $(t_{\partial \Sigma_{h,1}}, t_{\partial \Sigma_{g-h,1}}^{-1})$. Hence, we have

$$H_1(\mathcal{H}_g(c^{\text{ori}}); \mathbb{Z}) \cong (\mathbb{Z} \oplus \mathbb{Z}) / \langle (4h(2h+1), -4(g-h)(2g-2h+1)) \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}.$$

\square

Proof of Proposition 1.2 (ii). As explained in the paragraph before Lemma 3.8, $\mathcal{H}_{g,1}$ is generated by $t_{c_1}, \dots, t_{c_{2g}}$. Thus, $\mathcal{H}_g(c)$ is generated by $t_{c_1}, t_{c_2}, \dots, t_{c_{2h}}, t_{c_{2h+2}}, t_{c_{2h+3}}, \dots, t_{c_{2g+2}}$ by Lemma 3.7. \square

4. LOCALIZATION OF THE SIGNATURE OF DIRECTED BLFs

In this section, we compute the signature of $f^{-1}(A_i)$, and show that the signature of hyperelliptic directed BLF localizes. In Section 4.1 and Section 4.2, we calculate the signature of round cobordisms. In Section 4.3, we define a homomorphism $h_{g,c}$ for a simple closed curve c , and prove Proposition 1.3. In Section 4.4, we will prove Theorem 4.4.

4.1. Lemmas on round cobordisms. We use the notation in Introduction and Section 2.1. Let $f : M \rightarrow S^2$ be a directed BLF. In [4, Lemma 2.2], Baykur observed that M_i in a directed BLF is obtained by gluing a round 2-handle to a surface bundle over an annulus. We review his observation and investigate the signature of $f^{-1}(A_i)$.

Lemma 4.1 (Baykur [4]). *Let $f : M \rightarrow S^2$ be a directed BLF. Identifying $f^{-1}(r_i) \cap M_i$ with the surface Σ_{g_i} , consider the vanishing cycle d_i of Z_i is in Σ_{g_i} . Then, the monodromy φ of $f^{-1}(r_i) \cap M_i$ along $\partial_0 A_i$ is in $\mathcal{M}_{g_i}(d_i)$.*

Moreover, when d_i is a separating curve, the monodromy φ is in $\mathcal{M}_{g_i}(d_i^{\text{ori}})$. This is because, if φ changes the orientation of d_i , the monodromy along $\partial_1 A_i$ permutes the component of $f^{-1}(\gamma_{i+1})$. Inductively, the monodromy along $\partial_1 A_m$ permutes the component of $f^{-1}(\gamma_{m+1})$. However, since $f^{-1}(D_l)$ is a trivial surface bundle over a disk, it must be trivial.

In the paragraph after Lemma 3.5, we defined the homomorphisms $\Phi_n : \mathcal{M}_{g_i}(c) \rightarrow \mathcal{M}_{g_i-1}$ and $\Phi_s : \mathcal{M}_{g_i}(c^{\text{ori}}) \rightarrow \mathcal{M}_h \times \mathcal{M}_{g_i-h}$. Baykur [4] also observed that the monodromy of $f^{-1}(r_i) \cap M_i$ along

$\partial_1 A_i$ is $\Phi_n(\varphi)$ when d_i is non-separating and $\Phi_s(\varphi)$ when d_i is separating, for a suitable identification of $f^{-1}(r_{i+1}) \cap M_i$ with Σ_{g_i-1} if d_i is type I and with $\Sigma_h \cup \Sigma_{g_i-h}$ if d_i is type II_h.

For a mapping class $\varphi \in \mathcal{M}_g(c)$ represented by $T \in \text{Diff}_+ \Sigma_g$ satisfying $T(c) = c$, define a mapping torus V_φ by $V_\varphi = \Sigma_g \times [0, 1] / ((0, T(x)) \sim (1, x))$. We can identify $f^{-1}(\partial_0 A_i) \cap M_i$ with V_φ for some $\varphi \in \mathcal{M}_{g_i}(d_i)$. Identifying D^2 with the unit disk in \mathbf{C} , define an equivalence relation \sim_φ on $D^2 \times [-1, 1] \times [0, 1]$ by

$$\begin{cases} (v, s, 1) \sim (v, s, 0), & \text{if } \varphi \text{ preserves the orientation of } c, \\ (v, s, 1) \sim (\bar{v}, -s, 0), & \text{if } \varphi \text{ reverses the orientation of } c, \end{cases}$$

where \bar{v} is the complex conjugate of v . The compact 4-manifold R defined by $R = D^2 \times [-1, 1] \times [0, 1] / \sim_\varphi$ is called a round 2-handle. Choose an embedding $j : \partial D^2 \times [-1, 1] \times [0, 1] / \sim_\varphi \rightarrow V_\varphi$ such that $j(\partial D^2, s, 0) = c \times \{0\} \subset V_\varphi$ and $p_2 j(x, s, t) = t$, where p_2 is the projection to the second factor. Then, he observed:

Lemma 4.2 (Baykur [4]).

$$M_i \cong (V_\varphi \times [0, 1]) \cup R,$$

for some embedding j as above.

We remark that the isotopy class of the attaching map $j : \partial D^2 \times [-1, 1] \times [0, 1] / \sim_T \rightarrow V_\varphi \times \{0\}$ is unique if the genus g is greater than or equal to 2.

The signature of $f^{-1}(A_i)$ is calculated as follows. Since the components of $f^{-1}(A_i)$ except M_i are surface bundles over the annulus A_i , we have $\text{Sign } f^{-1}(A_i) = \text{Sign } M_i$. By Lemma 4.2, we have:

Lemma 4.3.

$$\text{Sign } f^{-1}(A_i) = \text{Sign}((V_\varphi \times [0, 1]) \cup R).$$

4.2. Wall's non-additivity formula. In [16], the second author defined a class function $m : \mathcal{M}_{g,2} \rightarrow \mathbb{Q}\mathbb{P}^1$. We review this function, and calculate the signature of the compact 4-manifold $(V_\varphi \times [0, 1]) \cup R$ in Section 4.1.

For a mapping class $\varphi = [T] \in \mathcal{M}_{g,2}$, let $V'_\varphi = \Sigma_{g,2} \times [0, 1] / ((0, T(x)) \sim (1, x))$ be its mapping torus. Choose points x_1 and x_2 in each boundary component of $\Sigma_{g,2}$, and define a continuous map by $l_i : S^1 \rightarrow V'_\varphi$ by $l_i(t) = (t, x_i)$ for $i = 1, 2$. Let ∂_1 and ∂_2 be the two boundary components of $\Sigma_{g,2}$. Denote by e_1, e_2, e_3 , and e_4 the homology classes $[l_1], [l_2], [\partial_1 \times \{0\}]$, and $[\partial_2 \times \{0\}]$, respectively. Then, for some $p, q \in \mathbb{Q}$, the set $\{e_1 + e_2, p(e_3 - e_4) + qe_1\}$ forms a basis of $\text{Ker}(H_1(\partial V'_\varphi; \mathbb{Q}) \rightarrow H_1(V'_\varphi; \mathbb{Q}))$. The element $[p : q] \in \mathbb{Q}\mathbb{P}^1$ is unique, and we can define a function $m : \mathcal{M}_{g,2} \rightarrow \mathbb{Q}\mathbb{P}^1$ by $m(\varphi) = [p : q]$. Since it satisfies $m(\varphi t_{\partial_1} t_{\partial_2}^{-1}) = m(\varphi)$, it induces the class function on $\mathcal{M}_g(c^{\text{ori}})$. For simplicity, we also denote it by $m : \mathcal{M}_g(c^{\text{ori}}) \rightarrow \mathbb{Q}\mathbb{P}^1$.

Define a map $s : \mathcal{M}_g(c) \rightarrow \mathbb{Z}$ by $s(\varphi) = \text{Sign}((V_\varphi \times [0, 1]) \cup R)$. We can write the signature $s(\varphi)$ with the function $m : \mathcal{M}_g(c^{\text{ori}}) \rightarrow \mathbb{Q}\mathbb{P}^1$ as follows:

Lemma 4.4. Let $\varphi \in \mathcal{M}_g(c)$. Then, we have

$$s(\varphi) = \begin{cases} \text{sign}(m(\varphi)), & \text{if } c \text{ is non-separating and } \varphi \text{ preserves the orientation of } c, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We apply Wall's nonadditivity Formula to the pasting of the round 2-handle. First, we review his formula. Let X_- , X_0 , and X_+ be compact 3-manifolds, and let Y_- and Y_+ be compact 4-manifolds such that

$$\partial X_- = \partial X_+ = \partial X_0 = Z, \quad \partial Y_- = X_- \cup X_0, \quad \partial Y_+ = X_+ \cup X_0.$$

We denote by Y and X the compact 4-manifold $Y = Y_- \cup Y_+$ and the space $X = X_- \cup X_0 \cup X_+$, respectively. Suppose that Y is oriented inducing orientations of Y_- and Y_+ . Orient the other manifolds so that

$$\partial_*[Y_-] = [X_0] - [X_-], \quad \partial_*[Y_+] = [X_+] - [X_0], \quad \partial_*[X_-] = \partial_*[X_+] = \partial_*[X_0] = [Z].$$

Let V , A , B , and C denote the vector spaces $V = H_1(Z; \mathbb{Q})$, $A = \text{Ker}(H_1(Z; \mathbb{Q}) \rightarrow H_1(X_-; \mathbb{Q}))$, $B = \text{Ker}(H_1(Z; \mathbb{Q}) \rightarrow H_1(X_0; \mathbb{Q}))$, and $C = \text{Ker}(H_1(Z; \mathbb{Q}) \rightarrow H_1(X_+; \mathbb{Q}))$. On the vector space $W = B \cap (C + A) / ((B \cap C) + (B \cap A))$, Wall defined a symmetric bilinear map $\Psi : W \times W \rightarrow \mathbb{Q}$ as follows. Let $I : H_1(Z; \mathbb{Q}) \times H_1(Z; \mathbb{Q}) \rightarrow \mathbb{Q}$ denote the intersection form, and $b, b' \in B \cap (C + A)$. Since $b' \in B \cap (C + A)$, there exist $c' \in C$ and $a' \in A$ such that $a' + b' + c' = 0$. Then, define a map $\Psi : W \times W \rightarrow \mathbb{Q}$ by $\Psi'([b], [b']) = I(b, c')$. He showed that this map is well-defined and symmetric. Denote by $\text{Sign}(V; B, C, A)$ the signature of this symmetric bilinear form. His signature formula is:

Theorem 4.5 (Wall [17]).

$$\text{Sign } Y = \text{Sign } Y_- + \text{Sign } Y_+ - \text{Sign}(V; B, C, A).$$

Next, we apply his formula to our settings. We should let Y_- and Y_+ denote the manifolds

$$Y_- = D^2 \times [-1, 1] \times [0, 1] / \sim \quad \text{and} \quad Y_+ = V_\varphi \times [0, 1],$$

respectively. The rest of the manifolds are

$$\partial Y_- = (\partial D^2 \times [-1, 1] \times [0, 1] / \sim) \cup (D^2 \times \{-1, 1\} \times [0, 1] / \sim),$$

$$\partial Y_+ = (V_\varphi \times \{1\}) \amalg (V_\varphi \times \{0\}),$$

$$X_0 = \partial D^2 \times [-1, 1] \times [0, 1] / \sim, \quad X_- = D^2 \times \{-1, 1\} \times [0, 1] / \sim,$$

$$X_+ = (V_\varphi \times \{1\}) \amalg (V_\varphi \times \{0\} - \text{Int } j(X_0)), \quad Z = \partial D^2 \times \{-1, 1\} \times [0, 1] / \sim.$$

Consider the case when $T|_{\nu(c)} = id$. Choose a point x in ∂D^2 . Define continuous maps $f_i : S^1 \rightarrow \partial D^2 \times \{-1, 1\} \times S^1$ by $f_i(t) = (x, (-1)^i, t)$ for $i = 1, 2$. Then, the set consisting of the homology classes $e_1 = [\partial D^2 \times \{-1\}]$, $e_2 = [\partial D^2 \times \{1\}]$, $e_3 = [f_1]$, and $e_4 = [f_2]$ in $H_1(Z; \mathbb{Q})$ forms a basis.

When c is separating, we have $A = C = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$. Hence, we obtain $W = (B \cap (C + A)) / ((B \cap C) + (B \cap A)) = 0$. When c is non-separating, $\text{Sign}(V_\varphi \times [0, 1] \cup R)$ is calculated in [16, Lemma 3.4].

Consider the case when $T|_{\nu(c)} = r$. In this case, the curve c is non-separating. Define a continuous map $f : S^1 \rightarrow \partial D^2 \times \{-1, 1\} \times [0, 1] / \sim$ by

$$f(t) = \begin{cases} (x, -1, 2t) & \text{when } 0 \leq t \leq \frac{1}{2}, \\ (x, 1, 2t - 1) & \text{when } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The set of homology classes consisting of $e_1 = [\partial D^2 \times \{-1\}]$ and $e_2 = [f]$ in $H_1(Z; \mathbb{Q})$ forms a basis. In this case, $A = B = \mathbb{Q}e_1$. Hence, we have $W = 0$.

□

4.3. The homomorphism $h_{g,c}$. Let c be a simple closed curve in Σ_g . Since the neighborhood $\nu(c)$ of c is diffeomorphic to $\partial D^2 \times [-1, 1]$, we obtain a manifold $L(c) = \Sigma_g \times [0, 1] \cup_{\nu(c)} (D^2 \times [-1, 1])$ by gluing $D^2 \times [-1, 1]$ along $\nu(c)$. This is diffeomorphic to a fiber of the projection $(V_\varphi \times [0, 1]) \cup R \rightarrow S^1$ to the last factor. We denote $\tilde{V}_\varphi = (V_\varphi \times [0, 1]) \cup R$ in the following.

Let φ and ψ be mapping classes in $\mathcal{M}_g(c)$. For example, by gluing the $L(c)$ -bundles $\tilde{V}_\varphi \times [0, 1]$ and $\tilde{V}_\psi \times [0, 1]$ on an annulus, we obtain a $L(c)$ -bundle over $S^2 - \Pi_{i=1}^3 \text{Int } D^2$ whose fiberwise boundary is $E_{\varphi,\psi} \amalg -E_{\Phi(\varphi),\Phi(\psi)}$ and the whole boundary is

$$(E_{\varphi,\psi} \amalg -E_{\Phi(\varphi),\Phi(\psi)}) \cup_{\partial E_{\varphi,\psi} \amalg -\partial E_{\Phi(\varphi),\Phi(\psi)}} (-\tilde{V}_\varphi \amalg -\tilde{V}_\psi \amalg -\tilde{V}_{(\varphi\psi)^{-1}}).$$

Hence, we have

$$\text{Sign } E_{\varphi,\psi} - \text{Sign } E_{\Phi(\varphi),\Phi(\psi)} - \text{Sign } \tilde{V}_\varphi - \text{Sign } \tilde{V}_\psi - \text{Sign } \tilde{V}_{(\varphi\psi)^{-1}} = 0.$$

If we rewrite it by Meyer's signature cocycle and the function $s : \mathcal{M}_g(c) \rightarrow \mathbb{Z}$, we have

$$\begin{cases} -\tau_g(\varphi, \psi) + \Phi^* \tau_{g-1}(\varphi, \psi) - \delta s(\varphi, \psi) = 0 \in C^2(\mathcal{M}_g(c); \mathbb{Z}) & \text{if } c \text{ is type I,} \\ -\tau_g(\varphi, \psi) + \Phi^*(\tau_h \times \tau_{g-h})(\varphi, \psi) - \delta s(\varphi, \psi) = 0 \in C^2(\mathcal{M}_g(c^{\text{ori}}); \mathbb{Z}) & \text{if } c \text{ is type II}_h. \end{cases}$$

If we restrict the Meyer cocycles to \mathcal{H}_g , we have $\tau_g = \delta \phi_g \in C^2(\mathcal{H}_g; \mathbb{Q})$, and $\tau_{g-1} = \delta \phi_{g-1} \in C^2(\mathcal{H}_{g-1}; \mathbb{Q})$. Thus, we have proved:

Lemma 4.6. *When c is type I, define a function $h_{g,c} : \mathcal{H}_g(c) \rightarrow \mathbb{Q}$ by*

$$h_{g,c}(\varphi) = s(\varphi) + \phi_g(\varphi) - \Phi^* \phi_{g-1}(\varphi).$$

When c is type II_h, define $h_{g,c} : \mathcal{H}_g(c^{\text{ori}}) \rightarrow \mathbb{Q}$ by

$$h_{g,c}(\varphi) = s(\varphi) + \phi_g(\varphi) - \Phi^*(\phi_h \times \phi_{g-h})(\varphi).$$

Then, both of these maps are homomorphisms.

Proof of Proposition 1.3. First, consider the case when the vanishing cycle c is type I in Figure 1. Since $h_{g,c}$ is a homomorphism, we have $h_{g,c}(\iota_g) = 0$. The mapping classes t_{c_i} for $i = 1, 2, \dots, 2g-1$ are mutually conjugate in $\mathcal{H}_g(c)$. Therefore, we have $h_{g,c}(t_{c_1}) = \dots = h_{g,c}(t_{c_{2g-1}})$. By the chain relation, we have $(t_{c_1} \cdots t_{c_{2g-1}})^{2g} = t_{c_{2g+1}}^2$. Thus, we obtain $h_{g,c}(t_{c_{2g+1}}) = g(2g-1)h_{g,c}(t_{c_1})$. Hence, it suffices to show that $h_{g,c}(\sigma_{2g+1}) = -g/(2g+1)$.

In [7, Lemma 3.3], Endo showed that $\phi_g(t_{c_{2g+1}}) = (g+1)/(2g+1)$. Since $\Phi(t_{c_{2g+1}}) = 1 \in \mathcal{M}_{g-1}$, we have $\Phi^* \phi_{g-1}(t_{c_{2g+1}}) = 0$. By Lemma 4.4, we have

$$s(t_{c_{2g+1}}) = \text{sign } m(t_{c_{2g+1}}) = \text{sign}([1 : -1]) = -1.$$

Thus, we obtain $h_{g,c}(t_{c_{2g+1}}) = -g/(2g+1)$.

Next, consider the case when the vanishing cycle c is type II in Figure 1. When $1 \leq h \leq g-1$, this follows from [7, Lemma 3.3] since $s(t_{c_i}) = 0$. When $h = 0$, $h_{g,c}$ is the zero map since $H^1(\mathcal{H}_g(c); \mathbf{Q}) = H^1(\mathcal{H}_g; \mathbf{Q}) = 0$. \square

4.4. Proof of Theorem 1.1. We prepare the hyperelliptic mapping class group of the non-connected surface $f^{-1}(r_i)$, where the monodromy of it along $\partial_0 A_i$ lies. Identify $f^{-1}(r_i)$ with some standard surface $S_i = \Sigma_{n_i(1)} \amalg \cdots \amalg \Sigma_{n_i(k_i)}$, where $n_i(1), \dots, n_i(k_i)$ are non-negative integers. We may assume that the action on $f^{-1}(r_i)$ induced by ι_g in Definition 2.4 coincides with $\iota_{n_i(1)} \amalg \cdots \amalg \iota_{n_i(k_i)}$, and the vanishing cycle d_i lies in $\Sigma_{n_i(1)}$ ($n_i(1) = g_i$). Define groups \mathcal{H}_{S_i} and $\mathcal{H}_{S_i}(d_i)$ by $\mathcal{H}_{S_i} = \mathcal{H}_{n_i(1)} \times \cdots \times \mathcal{H}_{n_i(k_i)}$ and $\mathcal{H}_{S_i}(d_i) = \mathcal{H}_{n_i(1)}(d_i) \times \mathcal{H}_{n_i(2)} \times \cdots \times \mathcal{H}_{n_i(k_i)}$, respectively. By Definition 2.4, the monodromy $\tilde{\varphi}_1$ along $\partial_0 A_1$ is contained in $\mathcal{H}_{S_1}(d_1)$. Denote by Φ the homomorphism Φ_n if d_1 is non-separating, and Φ_s if d_1 is separating. As stated in Section 4.1, the monodromy $\tilde{\varphi}_2$ of $f^{-1}(r_2)$ along $\partial_0 A_2$ is the image of $\tilde{\varphi}_1 \in \mathcal{H}_{S_1}(d_1)$ under $\Phi : \mathcal{H}_{S_1}(d_1) \rightarrow \mathcal{H}_{S_2}$. By Lemma 4.1, it is contained in $\mathcal{H}_{S_2}(d_2)$. Define a natural homomorphism $\Phi_{S_i} : \mathcal{H}_{S_i}(d_i) \rightarrow \mathcal{H}_{S_{i+1}}$ by $\Phi_{S_i}(x_1, x_2, \dots, x_{k_i}) = (\Phi(x_1), x_2, \dots, x_{k_i})$, for $i = 1, \dots, m$, where Φ denotes Φ_n if d_i is non-separating, and Φ_s if d_i is separating. Inductively, the monodromy $\tilde{\varphi}_i$ along $\partial_0 A_i$ is contained in $\mathcal{H}_{S_i}(d_i)$, and $\tilde{\varphi}_{i+1} = \Phi_{S_i}(\tilde{\varphi}_i)$.

By the Novikov additivity, we have

$$\begin{aligned} \text{Sign } M &= \sum_{i=1}^m \text{Sign } f^{-1}(A_i) + \text{Sign } f^{-1}(D_l) + \text{Sign } f^{-1}(D_h) \\ &= \sum_{i=1}^m \text{Sign } f^{-1}(A_i) + \text{Sign } f^{-1}(D_h) - \prod_{j=1}^n \text{Int } \nu(y_j) + \sum_{j=1}^n \text{Sign } f^{-1}(\nu(y_j)). \end{aligned}$$

Define the Meyer function $\phi_{S_i} : \mathcal{H}_{S_i} \rightarrow \mathbb{Q}$ by $\phi_{S_i}(x_1, \dots, x_{k_i}) = \sum_{j=1}^{k_i} \phi_{S_i}(x_j) \in C^1(\mathcal{H}_{S_i}; \mathbb{Q})$. Let $\psi_j \in \mathcal{H}_g$ denote the monodromy along the loop a_j around the image $y_i \in D_h$ of each Lefschetz singularity for $j = 1, \dots, n$ in Section 2.2. By Lemma 2.9 and Lemma 4.4, we have

$$\text{Sign } M = \sum_{i=1}^m s(\varphi_i) + \left(-\phi_g(\tilde{\varphi}_1^{-1}) - \sum_{j=1}^n \phi_g(\psi_j) \right) + \sum_{j=1}^n \text{Sign } f^{-1}(\nu(y_j)),$$

where φ_i is the monodromy of $f^{-1}(r_i) \cap M_i$ along $\partial_0 A_i$. Since $f^{-1}(D_l)$ is a trivial bundle, we have $\tilde{\varphi}_{m+1} = 1 \in \mathcal{H}_{S_{m+1}}$. Since $\Phi_{S_i}(\tilde{\varphi}_i) = \tilde{\varphi}_{i+1} \in \mathcal{H}_{S_{i+1}}(d_{i+1})$, we have

$$\sum_{i=1}^m (\phi_{S_i}(\tilde{\varphi}_i) - \Phi_{S_i}^* \phi_{S_{i+1}}(\tilde{\varphi}_i)) = \phi_g(\tilde{\varphi}_1).$$

Since the Meyer function has the property $\phi_g(\varphi^{-1}) = -\phi_g(\varphi)$ (see [7]) for any $\varphi \in \mathcal{H}_g$, we obtain

$$\text{Sign } M = \sum_{i=1}^m (s(\varphi_i) + \phi_{S_i}(\tilde{\varphi}_i) - \Phi_{S_i}^* \phi_{S_{i+1}}(\tilde{\varphi}_i)) + \sum_{j=1}^n (-\phi_g(\psi_j) + \text{Sign } f^{-1}(\nu(y_j))).$$

By the definition of Φ_{S_i} , we have

$$\begin{aligned} &\phi_{S_i}(x_1, \dots, x_{k_i}) - \Phi_{S_{i+1}}^* \phi_{S_{i+1}}(x_1, \dots, x_{k_{i+1}}) \\ &= \begin{cases} \phi_{g_i}(x_1) - \Phi_n^* \phi_{g_i}(x_1), & \text{if } d_i \text{ is nonseparating,} \\ \phi_{g_i}(x_1) - \Phi_s^*(\phi_h \times \phi_{g_i-h})(x_1), & \text{if } d_i \text{ bounds subsurfaces of genus } h \text{ and } g_i - h. \end{cases} \end{aligned}$$

Thus, we have

$$\text{Sign } M = \sum_{i=1}^m h_{g_i, d_i}(\varphi_i) + \sum_{j=1}^n \sigma_{\text{loc}}(f^{-1}(y_j)).$$

5. EXAMPLES

Let $c_1, \dots, c_n \subset \Sigma_g$ be simple closed curves described in Figure 4.

Example 5.1. As shown in the proof of [9, Theorem 1.4], there exists a simplified BLF $f_{g,n} : M_{g,n} \rightarrow S^2$ which has the following Hurwitz system:

$$(t_{c_{2g}} \cdots t_{c_2} t_{c_1}^2 t_{c_2} \cdots t_{c_{2g}})^{2n},$$

and the vanishing cycle of the indefinite fold is c_{2g+1} .

By the definition of $f_{g,n}$, it is hyperelliptic. We denote by $y_1, \dots, y_{8gn} \in S^2$ the critical values of $f_{g,n}$. By using the formula in Theorem 1.1, the signature of $M_{g,n}$ can be calculated as follows:

$$\begin{aligned} \text{Sign } M_{g,n} &= \sum_{i=1}^{8gn} \sigma_{\text{loc}}(f_{g,n}^{-1}(y_i)) + h((t_{c_{2g}} \cdots t_{c_2} t_{c_1}^2 t_{c_2} \cdots t_{c_{2g}})^{2n}) \\ &= 8gn \cdot \frac{-g-1}{2g+1} + h(t_{c_{2g+1}}^{-4n}) \\ &= \frac{-8g^2n - 8gn}{2g+1} + (-4n) \cdot \frac{-g}{2g+1} \\ &= -4gn. \end{aligned}$$

It is easy to see that $M_{g,n}$ is simply connected and that the Euler characteristic of $M_{g,n}$ is $8gn - 4g + 6$. As shown in [9], $M_{g,n}$ is spin if and only if both of the integers g and n are even. Thus, by Freedman's theorem, $M_{g,n}$ is homeomorphic to $\# \frac{gn}{4} E(2) \# (\frac{5gn}{4} - 2g + 2) S^2 \times S^2$ if both g and n are even and $\#(2gn - 2g + 2) \mathbb{CP}^2 \# (6gn - 2g + 2) \mathbb{CP}^2$ otherwise.

Example 5.2. As shown in the proof of [9, Theorem 1.4], there exists a simplified BLF $\tilde{f}_{g,n} : \tilde{M}_{g,n} \rightarrow S^2$ which has the following Hurwitz system:

$$(t_{c_{2g}} \cdots t_{c_2} t_{c_1}^2 t_{c_2} \cdots t_{c_{2g}})^{2n} \cdot (t_{c_1} \cdots t_{c_{2g-2}})^{2(2g-1)n},$$

and a vanishing cycle of the indefinite fold is c_{2g+1} .

By the definition of $\tilde{f}_{g,n}$, it is hyperelliptic. We denote by $\tilde{y}_1, \dots, \tilde{y}_{8g^2n-4gn+4n} \in S^2$ the critical values of $\tilde{f}_{g,n}$. By using the formula in Theorem 1.1, the signature of $\tilde{M}_{g,n}$ can be calculated as follows:

$$\begin{aligned} \text{Sign } \tilde{M}_{g,n} &= \sum_{i=1}^{8g^2n-4gn+4n} \sigma_{\text{loc}}(\tilde{f}_{g,n}^{-1}(\tilde{p}_i)) + h((t_{c_{2g}} \cdots t_{c_2} t_{c_1}^2 t_{c_2} \cdots t_{c_{2g}})^{2n} \cdot (t_{c_1} \cdots t_{c_{2g-2}})^{2(2g-1)n}) \\ &= (8g^2n - 4gn + 4n) \cdot \frac{-g-1}{2g+1} + 2n \cdot h(t_{c_{2g+1}}^{-2} \cdot t_g) + 2(2g-1)n \cdot h(t_{c_1} \cdots t_{c_{2g-2}}) \\ &= \frac{-8g^3n + 4g^2n - 4gn - 8g^2n + 4gn - 4n}{2g+1} - 4n \cdot \frac{-g}{2g+1} + 2(2g-1)n(2g-2) \cdot \frac{-1}{4g^2-1} \\ &= -4g^2n. \end{aligned}$$

It is easy to see that $\tilde{M}_{g,n}$ is simply connected, and that the Euler characteristic of $\tilde{M}_{g,n}$ is $8g^2n - 4gn + 4n - 4g + 6$. As shown in [9], $\tilde{M}_{g,n}$ is spin if and only if g is even. Thus, we can easily determine the homeomorphism type of $\tilde{M}_{g,n}$ as in Example 5.1.

Acknowledgments. The authors would like to express their gratitude to Hisaaki Endo for his continuous support during the course of this work, Nariya Kawazumi for his helpful comments for the draft of this paper. The first author is supported by Yoshida Scholarship 'Master21' and he is grateful to Yoshida Scholarship Foundation for their support. The second author is supported by JSPS Research Fellowships for Young Scientists (22-2364).

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